数列の和（2021年北大）

(1) $a_1 = S_1 = \frac{1}{6} \cdot 1 \cdot 2 \cdot 9 = 3$

$n \geqq 2$ のとき $a_n = S_n - S_{n-1} = \frac{1}{6} n(n+1)(2n+7) - \frac{1}{6}(n-1)n(2n+5)$

$= \frac{1}{6} n(2n^2+9n+7-2n^2-3n+5) = \frac{1}{6} n(6n+12) = n(n+2)$

これは $n=1$ のときも成り立つ。 $\therefore a_n = n(n+2)$

(2) $n=1$ のとき $\sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{a_1} = \frac{1}{3}$

$n \geqq 2$ のとき $\sum_{k=1}^{n} \frac{1}{a_k} = \sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+2} \right)$

$= \frac{1}{2} \left\{ \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{n-2}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right) \right\}$

$= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right)$

$= \frac{1}{2} \cdot \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{2(n+1)(n+2)}$

$= \frac{3n^2+9n+6-2n-4-2n-2}{4(n+1)(n+2)}$

$= \frac{n(3n+5)}{4(n+1)(n+2)}$

これは $n=1$ のときも成り立つ。 $\therefore \sum_{k=1}^{n} \frac{1}{a_k} = \frac{n(3n+5)}{4(n+1)(n+2)}$

（補足） (2)の答案を書く前に、以下のように実験して、$n=1$ のときと $n \geqq 2$ のときに分ける必要性に気づきたい。

$n=1$ のとき $\sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{a_1} = \frac{1}{3}$

$n=2$ のとき $\sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{a_1} + \frac{1}{a_2} = \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} = \frac{1}{2} \left\{ \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) \right\}$

$n=3$ のとき $\sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} = \frac{1}{2} \left\{ \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) \right\}$

$n=4$ のとき $\sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{2} \left\{ \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) \right\}$

$n=5$ のとき $\sum_{k=1}^{n} \frac{1}{a_k} = \frac{1}{2} \left\{ \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \left(\cancel{\frac{1}{5}} - \frac{1}{7}\right) \right\}$